

# NATURAL CONNECTION WITH TOTALLY SKEW-SYMMETRIC TORSION ON RIEMANNIAN ALMOST PRODUCT MANIFOLDS

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On a Riemannian almost product manifold  $(M, P, g)$  we consider a linear connection preserving the almost product structure  $P$  and the Riemannian metric  $g$  and having a totally skew-symmetric torsion. We determine the class of the manifolds  $(M, P, g)$  admitting such a connection and prove that this connection is unique in terms of the covariant derivative of  $P$  with respect to the Levi-Civita connection. We find a necessary and sufficient condition the curvature tensor of the considered connection to have similar properties like the ones of the Kähler tensor in Hermitian geometry. We pay attention to the case when the torsion of the connection is parallel. We consider this connection on a Riemannian almost product manifold  $(G, P, g)$  constructed by a Lie group  $G$ .

*Keywords:* Riemannian manifold; almost product structure; non-integrable structure; linear connection; Bismut connection; KT-connection; totally skew-symmetric torsion; parallel torsion; Lie group; Killing metric.

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## Introduction

There is a strong interest in the metric connections with totally skew-symmetric torsion (3-form). These connections arise in a natural way in theoretical and mathematical physics. For example, such a connection is of particular interest in string theory [1]. In mathematics this connection was used by Bismut to prove a local index theorem for non-Kähler Hermitian manifolds [2]. Such a connection is known as a *KT-connection* (Kähler with torsion) or a *Bismut connection* on an almost Hermitian manifold. The KT-geometry is a natural generalization of the Kähler geometry, since when the torsion is zero the KT-connection coincides with the Levi-Civita connection. According to Gauduchon [3], on any Hermitian manifold, there exists a unique connection preserving the almost complex structure and the metric, whose torsion is totally skew-symmetric.

There are lots of physical applications involving a Riemannian almost product manifold  $(M, P, g)$ , *i.e.* a differentiable manifold  $M$  with an almost product structure  $P$  and a Riemannian metric  $g$ . The goal of the present work is to study the linear connections on  $(M, P, g)$  with totally skew-symmetric torsion preserving the structures  $P$  and  $g$ . We use the notions of a *Riemannian P-manifold*, a *Riemannian*

*P-tensor* and a *RPT-connection* (Riemannian *P*- with torsion) as analogues of the notions of a Kähler tensor, a Kähler manifold and a KT-connection in Hermitian geometry. The point in question in this work is the determination of the class of manifolds admitting a RPT-connection, the uniqueness of the RPT-connection on such manifolds and the investigation of an example of such a connection.

The present paper is organized as follows. In Sec. 1 we give necessary facts about Riemannian almost product manifolds. We recall facts about natural connections (*i.e.* connections preserving *P* and *g*) with torsion on Riemannian almost product manifolds. The main results are Theorem 5 and Theorem 6 in Sec. 2. We prove that a RPT-connection exists only on a Riemannian almost product manifold in the class  $\mathcal{W}_3$  from the classification in [4]. This classification is made regarding the covariant derivatives of *P* with respect to the Levi-Civita connection  $\nabla$ . We establish the presence of a unique RPT-connection  $\nabla'$  which torsion is expressed by  $\nabla P$ . We find a relation between the scalar curvatures for  $\nabla$  and  $\nabla'$  and prove that they are equal if and only if  $(M, P, g)$  is a Riemannian *P*-manifold. In Sec. 3 we obtain a relation between the curvature tensors of  $\nabla$  and  $\nabla'$ . It is a necessary and sufficient condition the curvature tensor of  $\nabla'$  to be a Riemannian *P*-tensor. In Sec. 4 we consider the case when the torsion of  $\nabla'$  is parallel and its curvature tensor is a Riemannian *P*-tensor. In Sec. 5 we study the RPT-connection  $\nabla'$  on an example of a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(G, P, g)$  constructed in [5] by a Lie group  $G$ .

## 1. Preliminaries

Let  $(M, P, g)$  be a *Riemannian almost product manifold*, *i.e.* a differentiable manifold  $M$  with a tensor field *P* of type  $(1, 1)$  and a Riemannian metric *g* such that

$$P^2x = x, \quad g(Px, Py) = g(x, y)$$

for arbitrary  $x, y$  of the algebra  $\mathfrak{X}(M)$  of the smooth vector fields on  $M$ . Obviously  $g(Px, y) = g(x, Py)$  is valid.

Further  $x, y, z, w$  will stand for arbitrary elements of  $\mathfrak{X}(M)$  or vectors in the tangent space  $T_p M$  at  $p \in M$ .

In this work we consider Riemannian almost product manifolds with  $\text{tr}P = 0$ . In this case  $(M, P, g)$  is an even-dimensional manifold. If  $\dim M = 2n$  then the *associated metric*  $\tilde{g}$  of *g*, determined by  $\tilde{g}(x, y) = g(x, Py)$ , is an indefinite metric of signature  $(n, n)$ .

The classification in [4] of Riemannian almost product manifolds is made with respect to the tensor *F* of type  $(0, 3)$ , defined by

$$F(x, y, z) = g((\nabla_x P)y, z), \quad (1)$$

where  $\nabla$  is the Levi-Civita connection of *g*. The tensor *F* has the following properties:

$$F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \quad F(x, y, Pz) = -F(x, Py, z). \quad (2)$$

The basic classes of the classification in [4] are  $\mathcal{W}_1$ ,  $\mathcal{W}_2$  and  $\mathcal{W}_3$ . Their intersection is the class  $\mathcal{W}_0$ , determined by the condition  $F = 0$  or equivalently  $\nabla P = 0$ . A manifold  $(M, P, g)$  with  $\nabla P = 0$  is called a *Riemannian P-manifold* in [6]. In this classification there are included the classes  $\mathcal{W}_1 \oplus \mathcal{W}_2$ ,  $\mathcal{W}_1 \oplus \mathcal{W}_3$ ,  $\mathcal{W}_2 \oplus \mathcal{W}_3$  and the class  $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$  of all Riemannian almost product manifolds.

In the present work we consider manifolds of the class  $\mathcal{W}_3$ . This class is determined by the condition

$$\mathfrak{S}_{x,y,z} F(x, y, z) = 0, \quad (3)$$

where  $\mathfrak{S}_{x,y,z}$  is the cyclic sum by  $x, y, z$ . This is the only class of the basic classes  $\mathcal{W}_1$ ,  $\mathcal{W}_2$  and  $\mathcal{W}_3$ , where each manifold (which is not a Riemannian  $P$ -manifold) has a non-integrable almost product structure  $P$ . This means that in  $\mathcal{W}_3$  the Nijenhuis tensor  $N$ , determined by

$$N(x, y) = (\nabla_x P) Py - (\nabla_y P) Px + (\nabla_{Px} P) y - (\nabla_{Py} P) x,$$

is non-zero.

Further, the manifolds of the class  $\mathcal{W}_3$  we call *Riemannian almost product  $\mathcal{W}_3$ -manifolds*.

A tensor  $L$  of type (0,4) with properties

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z), \quad (4)$$

$$\mathfrak{S}_{x,y,z} L(x, y, z, w) = 0, \quad (5)$$

$$L(x, y, Pz, Pw) = L(x, y, z, w) \quad (6)$$

is called a *Riemannian P-tensor* [7].

As it is known the curvature tensor  $R$  of a Riemannian manifold with metric  $g$  is determined by  $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x,y]} z$  and the corresponding tensor of type (0,4) is defined as follows  $R(x, y, z, w) = g(R(x, y)z, w)$ . If the curvature tensor  $R$  on a Riemannian manifold  $(M, P, g)$  is a Riemannian  $P$ -tensor, then  $(M, P, g)$  is a Riemannian  $P$ -manifold.

Let the components of the inverse matrix of  $g$  with respect to the basis  $\{e_i\}$  of  $T_p M$  be  $g^{ij}$ . Then the quantities  $\rho$  and  $\tau$ , determined by  $\rho(y, z) = g^{ij}R(e_i, y, z, e_j)$  and  $\tau = g^{ij}\rho(e_i, e_j)$ , are the Ricci tensor and the scalar curvature for  $\nabla$ , respectively.

The *square norm* of  $\nabla P$  is defined by

$$\|\nabla P\|^2 = g^{ij}g^{ks}g((\nabla_{e_i} P)e_k, (\nabla_{e_j} P)e_s). \quad (7)$$

Obviously,  $\|\nabla P\|^2 = 0$  is valid if and only if  $(M, P, g)$  is a Riemannian  $P$ -manifold.

Let  $\nabla'$  be a linear connection with a tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  and a torsion  $T$ , i.e.

$$\nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]. \quad (8)$$

The corresponding (0,3)-tensors are defined by

$$Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z).$$

The symmetry of the Levi-Civita connection implies

$$T(x, y) = Q(x, y) - Q(y, x), \quad T(x, y) = -T(y, x).$$

A decomposition of the space  $\mathcal{T}$  of the torsion tensors  $T$  of type (0,3) (i.e.  $T(x, y, z) = -T(y, x, z)$ ) is valid on a Riemannian almost product manifold  $(M, P, g)$ :  $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4$ , where  $\mathcal{T}_i$  ( $i = 1, 2, 3, 4$ ) are the invariant orthogonal subspaces [8]. The projection operators  $p_i$  of  $\mathcal{T}$  in  $\mathcal{T}_i$  are determined as follows:

$$\begin{aligned} p_1(x, y, z) &= \frac{1}{8} \{ 2T(x, y, z) - T(y, z, x) - T(z, x, y) - T(Pz, x, Py) \\ &\quad + T(Py, z, Px) + T(z, Px, Py) - 2T(Px, Py, z) \\ &\quad + T(Py, Pz, x) + T(Pz, Px, y) - T(y, Pz, Px) \}, \\ p_2(x, y, z) &= \frac{1}{8} \{ 2T(x, y, z) + T(y, z, x) + T(z, x, y) + T(Pz, x, Py) \\ &\quad - T(Py, z, Px) - T(z, Px, Py) - 2T(Px, Py, z) \\ &\quad - T(Py, Pz, x) - T(Pz, Px, y) + T(y, Pz, Px) \}, \\ p_3(x, y, z) &= \frac{1}{4} \{ T(x, y, z) + T(Px, Py, z) - T(Px, y, Pz) - T(x, Py, Pz) \}, \\ p_4(x, y, z) &= \frac{1}{4} \{ T(x, y, z) + T(Px, Py, z) + T(Px, y, Pz) + T(x, Py, Pz) \}. \end{aligned} \tag{9}$$

**Definition 1** ([8]). A linear connection  $\nabla'$  on a Riemannian almost product manifold  $(M, P, g)$  is called a natural connection if  $\nabla'P = \nabla'g = 0$ .

If  $\nabla'$  is a linear connection with a tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  on a Riemannian almost product manifold, then it is a natural connection if and only if the following conditions are satisfied [8]:

$$\begin{aligned} F(x, y, z) &= Q(x, y, Pz) - Q(x, Py, z), \\ Q(x, y, z) &= -Q(x, z, y). \end{aligned} \tag{10}$$

Let  $\Phi$  be the (0,3)-tensor determined by

$$\Phi(x, y, z) = g(\tilde{\nabla}_x y - \nabla_x y, z),$$

where  $\tilde{\nabla}$  is the Levi-Civita connection of the associated metric  $\tilde{g}$ .

**Theorem 2** ([8]). A linear connection with torsion  $T$  on a Riemannian almost product manifold  $(M, P, g)$  is natural if and only if

$$\begin{aligned} 4p_1(x, y, z) &= -\Phi(x, y, z) + \Phi(y, z, x) - \Phi(x, Py, Pz) \\ &\quad - \Phi(y, Pz, Px) + 2\Phi(z, Px, Py), \end{aligned}$$

$$4p_3(x, y, z) = -g(N(x, y), z) = -2 \{ \Phi(z, Px, Py) + \Phi(z, x, y) \}.$$

□

**Theorem 3 ([5]).** *For the torsion  $T$  of a natural connection on a Riemannian  $\mathcal{W}_3$ -manifold  $(M, P, g) \notin \mathcal{W}_0$  the following properties hold*

$$p_1 = 0, \quad p_3 \neq 0.$$

□

## 2. RPT-Connection on a Riemannian Almost Product Manifold

Let  $\nabla'$  be a metric connection with totally skew-symmetric torsion  $T$  on a Riemannian manifold  $(M, g)$ . Since  $T$  is a 3-form, i.e.

$$T(x, y, z) = -T(y, x, z) = -T(x, z, y) = -T(z, y, x),$$

it is valid

$$Q(x, y, z) = \frac{1}{2}T(x, y, z). \quad (11)$$

Then we have

$$g(\nabla'_x y, z) = g(\nabla_x y, z) + \frac{1}{2}T(x, y, z).$$

The curvature tensors  $R$  and  $R'$  of  $\nabla$  and  $\nabla'$  and the corresponding Ricci tensors  $\rho$  and  $\rho'$  are related via the formulae (see e.g. [9,10]):

$$\begin{aligned} R(x, y, z, w) &= R'(x, y, z, w) - \frac{1}{2}(\nabla'_x T)(y, z, w) + \frac{1}{2}(\nabla'_y T)(x, z, w) \\ &\quad - \frac{1}{4}g(T(x, y), T(z, w)) - \frac{1}{4}\sigma^T(x, y, z, w), \end{aligned} \quad (12)$$

$$\rho(y, z) = \rho'(y, z) - \frac{1}{2}g^{ij}(\nabla'_{e_i} T)(y, z, e_j) - \frac{1}{4}g^{ij}g(T(e_i, y), T(z, e_j)), \quad (13)$$

where  $\sigma^T$  is the 4-form determined by

$$\sigma^T(x, y, z, w) = \sum_{x,y,z} g(T(x, y), T(z, w)). \quad (14)$$

Since  $\sigma^T$  and  $T$  are forms then the scalar curvatures  $\tau$  and  $\tau'$  for  $\nabla$  and  $\nabla'$  satisfy the following relation

$$\tau = \tau' - \frac{1}{4}g^{ij}g^{ks}g(T(e_i, e_k), T(e_s, e_j)). \quad (15)$$

**Definition 4.** *A natural connection with totally skew-symmetric torsion on a Riemannian almost product manifold  $(M, P, g)$  is called a RPT-connection.*

**Theorem 5.** *If a Riemannian almost product manifold  $(M, P, g)$  admits a RPT-connection then  $(M, P, g)$  is a Riemannian almost product  $\mathcal{W}_3$ -manifold.*

**Proof.** Let  $\nabla'$  is a RPT-connection on a Riemannian almost product manifold  $(M, P, g)$ . Since  $\nabla'$  is a natural connection then the tensor  $Q$  of the transformation  $\nabla \rightarrow \nabla'$  satisfies (10), which implies

$$\mathfrak{S}_{x,y,z} F(x, y, z) = \mathfrak{S}_{x,y,z} \{Q(x, y, Pz) - Q(x, Py, z)\}. \quad (16)$$

According to (11), the tensor  $Q$  is also 3-form and then we have

$$Q(x, y, Pz) = Q(y, Pz, x). \quad (17)$$

We apply (17) to (16) and obtain the characteristic condition (3) for the class  $\mathcal{W}_3$ .  $\square$

From (10), (11) and (3) we obtain the following properties of the torsion  $T$  of a RPT-connection:

$$\begin{aligned} T(x, y, z) &= T(Px, Py, z) - 2F(z, y, Px) \\ &= T(Px, y, Pz) - 2F(y, x, Pz) = T(x, Py, Pz) - 2F(x, Py, z). \end{aligned} \quad (18)$$

Let us suppose that  $(M, P, g)$  is not a Riemannian  $P$ -manifold. The condition  $(M, P, g) \notin \mathcal{W}_0$ , Theorem 3, (9) and (18) imply the following properties:

$$\begin{aligned} p_1(x, y, z) &= 0, \\ p_2(x, y, z) &= F(z, x, Py) \neq 0, \\ p_3(x, y, z) &= \frac{1}{2} \{F(x, y, Pz) + F(y, z, Px) - F(z, x, Py)\} \neq 0, \\ p_4(x, y, z) &= T(x, y, z) - \frac{1}{2} \mathfrak{S}_{x,y,z} F(x, y, Pz). \end{aligned} \quad (19)$$

**Theorem 6.** *On any Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g) \notin \mathcal{W}_0$  there exists a unique RPT-connection  $\nabla'$  which torsion is expressed by the tensor  $F$ . The following properties are valid for the torsion  $T$  of  $\nabla'$ :*

$$T \in \mathcal{T}_2 \oplus \mathcal{T}_3, \quad T \notin \mathcal{T}_2, \quad T \notin \mathcal{T}_3,$$

$$T(x, y, z) = \frac{1}{2} \mathfrak{S}_{x,y,z} F(x, y, Pz). \quad (20)$$

**Proof.** By (19) we establish directly that a RPT-connection on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g) \notin \mathcal{W}_0$  has a torsion  $T = p_2 + p_3 + p_4$ , i.e.  $T \in \mathcal{T}_2 \oplus \mathcal{T}_3 \oplus \mathcal{T}_4$ . Let us suppose that  $T = p_2 + p_4$ . Then (19) and (3) imply  $F = 0$ , i.e.  $(M, P, g) \in \mathcal{W}_0$  which is a contradiction. Therefore  $T \neq p_2 + p_4$  holds. Analogously we establish that  $T \neq p_3 + p_4$ . Consequently we obtain  $T \notin \mathcal{T}_2 \oplus \mathcal{T}_4$  and  $T \notin \mathcal{T}_3 \oplus \mathcal{T}_4$ . The condition  $T \in \mathcal{T}_2 \oplus \mathcal{T}_3$ , i.e.  $T = p_2 + p_3$ , according to (19) implies  $p_4 = 0$ , which is (20).  $\square$

Let  $\nabla'$  is the RPT-connection with torsion  $T$  determined by (20). Then (11), (1) and (3) imply

$$Q(x, y, z) = -\frac{1}{4} \{F(x, Py, z) - F(Px, y, z) - 2F(y, Px, z)\}, \quad (21)$$

$$Q(x, y) = -\frac{1}{4} \{(\nabla_x P) Py - (\nabla_{Px} P) y - 2(\nabla_y P) Px\}. \quad (22)$$

Bearing in mind (11) and (22), we have

$$T(x, y) = -\frac{1}{2} \{(\nabla_x P) Py - (\nabla_{Px} P) y - 2(\nabla_y P) Px\}.$$

According to (15), (7) and the latter equality, we obtain the following

**Proposition 7.** *The scalar curvatures  $\tau$  and  $\tau'$  for the connections  $\nabla$  and  $\nabla'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  are related via the formula*

$$\tau = \tau' + \frac{3}{8} \|\nabla P\|^2.$$

□

**Corollary 8.** *The scalar curvatures for the connections  $\nabla$  and  $\nabla'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  are equal if and only if  $(M, P, g)$  is a Riemannian  $P$ -manifold.* □

### 2.1. A relation of the RPT-connection with other connections

Now we will find the relation between the RPT-connection  $\nabla'$  and other two characteristic natural connections on a Riemannian almost product  $\mathcal{W}_3$ -manifold.

The canonical connection  $\nabla^C$  on a Riemannian almost product manifold  $(M, P, g)$  is a natural connection introduced in [8] as an analogue of the Hermitian connection in Hermitian geometry ([11,12,13]). This connection on  $(M, P, g) \in \mathcal{W}_3$  is studied in [5], where for the tensor  $Q^C$  of the transformation  $\nabla \rightarrow \nabla^C$  it is obtained

$$Q^C(x, y, z) = -\frac{1}{4} \{F(y, Px, z) - F(Py, x, z) + 2F(x, Py, z)\}. \quad (23)$$

The  $P$ -connection  $\nabla^P$  on a Riemannian almost product manifold  $(M, P, g)$  is a natural connection introduced in [7] as an analogue of the first canonical connection of Lichnerowicz in Hermitian geometry ([3,14,15]). In [7] this connection is studied on  $(M, P, g) \in \mathcal{W}_3$ , where for the tensor  $Q^P$  of the transformation  $\nabla \rightarrow \nabla^P$  it is obtained

$$Q^P(x, y, z) = -\frac{1}{2}F(x, Py, z). \quad (24)$$

Bearing in mind (21), (23), (24), (2) and (3), we get  $Q^P = \frac{1}{2}(Q^C + Q)$  and therefore it is valid the following

**Proposition 9.** *The P-connection  $\nabla^P$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  is the average connection of the canonical connection  $\nabla^C$  and the RPT-connection  $\nabla'$ , i.e.*

$$\nabla^P = \frac{1}{2} (\nabla^C + \nabla').$$

□

### 3. RPT-Connection with Riemannian P-Tensor of Curvature

In the present section we discuss the case when the curvature tensor  $R'$  of the RPT-connection  $\nabla'$  is a Riemannian P-tensor, i.e.  $R'$  has properties (4)–(6).

Since the torsion  $T$  of  $\nabla'$  and the quantity  $\sigma^T$  from (14) are forms, then by virtue of (12) it follows that  $R'$  satisfies (4). According to  $\nabla'P = 0$ , property (6) is valid, too. Then,  $R'$  is a Riemannian P-tensor if and only if (5) holds for  $R'$ .

Bearing in mind that (5) is satisfied for  $R$ , moreover  $T$  and  $\sigma^T$  are forms, then the following property is valid for a metric connection  $\nabla'$  with totally skew-symmetric torsion  $T$  on a Riemannian manifold  $(M, g)$ :

$$\mathfrak{S}_{x,y,z} R'(x, y, z, w) = \mathfrak{S}_{x,y,z} (\nabla'_x T)(y, z, w) + \sigma^T(x, y, z, w). \quad (25)$$

Then, condition (5) for  $R'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  holds if and only if

$$\mathfrak{S}_{x,y,z} (\nabla'_x T)(y, z, w) + \sigma^T(x, y, z, w) = 0. \quad (26)$$

**Theorem 10.** *The RPT-connection  $\nabla'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  has a Riemannian P-tensor of curvature  $R'$  if and only if*

$$R(x, y, z, w) = R'(x, y, z, w) - \frac{1}{4}g(T(x, y), T(z, w)) + \frac{1}{12}\sigma^T(x, y, z, w). \quad (27)$$

**Proof.** Let the RPT-connection  $\nabla'$  on  $(M, P, g)$  have a Riemannian P-tensor of curvature  $R'$ , i.e. (26) is valid. Since  $T$  and  $\sigma^T$  are forms, then the cyclic sum of (26) by  $y, z, w$  implies

$$\begin{aligned} 3(\nabla'_x T)(y, z, w) + 2(\nabla'_z T)(x, y, w) + 2(\nabla'_y T)(z, x, w) \\ - 2(\nabla'_w T)(x, y, z) + 3\sigma^T(x, y, z, w) = 0. \end{aligned} \quad (28)$$

From (26) and (28) it follows immediately

$$(\nabla'_x T)(y, z, w) - 2(\nabla'_w T)(x, y, z) + \sigma^T(x, y, z, w) = 0,$$

where by virtue of the substitution  $x \leftrightarrow w$  we have

$$(\nabla'_w T)(x, y, z) - 2(\nabla'_x T)(y, z, w) - \sigma^T(x, y, z, w) = 0.$$

The latter two equalities imply

$$(\nabla'_x T)(y, z, w) = -\frac{1}{3}\sigma^T(x, y, z, w) \quad (29)$$

and then, according to (12), we obtain (27).

*Vice versa,* let (27) be satisfied. Then, bearing in mind the equalities  $\mathfrak{S}_{x,y,z} R(x,y,z,w) = 0$  and  $\mathfrak{S}_{x,y,z} \sigma^T(x,y,z,w) = 3\sigma^T(x,y,z,w)$  for the 4-form  $\sigma^T$ , we establish that the cyclic sum of (27) by  $x, y, z$  yields  $\mathfrak{S}_{x,y,z} R'(x,y,z,w) = 0$ . Therefore,  $R'$  is a Riemannian  $P$ -tensor.  $\square$

Since condition (29) is fulfilled for a Riemannian  $P$ -tensor  $R'$ , then we have  $g^{ij} (\nabla'_{e_i} T)(y, z, e_j) = 0$ . Hence, because of (13), we obtain

**Corollary 11.** *If the RPT-connection  $\nabla'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  has a Riemannian  $P$ -tensor of curvature  $R'$ , then the Ricci tensors  $\rho$  and  $\rho'$  for  $\nabla$  and  $\nabla'$  are related via the formula*

$$\rho(y, z) = \rho'(y, z) - \frac{1}{4} g^{ij} g(T(e_i, y), T(z, e_j)).$$

$\square$

#### 4. RPT-Connection with Riemannian $P$ -Tensor of Curvature and Parallel Torsion

Let  $\nabla'$  be a metric connection with totally skew-symmetric torsion  $T$  on a Riemannian manifold  $(M, g)$ . If  $T$  is parallel, i.e.  $\nabla' T = 0$ , then (12) implies

$$R(x, y, z, w) = R'(x, y, z, w) - \frac{1}{4} g(T(x, y), T(z, w)) - \frac{1}{4} \sigma^T(x, y, z, w). \quad (30)$$

*Vice versa,* if (30) is valid, because of (12) and the totally skew-symmetry of  $T$ , then  $\nabla' T = 0$  holds. Therefore, relation (30) is a necessary and sufficient condition for a parallel torsion of  $\nabla'$ .

Because of  $\sigma^T(x, y, z, w) = \sigma^T(z, w, x, y)$  and  $R(x, y, z, w) = R(z, w, x, y)$ , relation (30) yields

$$R'(x, y, z, w) = R'(z, w, x, y). \quad (31)$$

Besides, if  $\nabla' T = 0$  then (25) implies

$$\mathfrak{S}_{x,y,z} R'(x, y, z, w) = \sigma^T(x, y, z, w). \quad (32)$$

Now, let  $\nabla'$  be the RPT-connection on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$ . Let us suppose that the torsion of  $\nabla'$  is parallel. Then (31) is valid. Moreover, (6) holds because of  $\nabla' P = 0$ . Therefore, (31) is equivalent to

$$R'(Px, Py, Pz, Pw) = R'(x, y, z, w). \quad (33)$$

Equalities (32) and (33) imply

$$\sigma^T(Px, Py, Pz, Pw) = \sigma^T(x, y, z, w).$$

In addition to the above, let us suppose that the curvature tensor  $R'$  of the RPT-connection  $\nabla'$  is a Riemannian  $P$ -tensor. Then (29) is valid and because of  $\nabla' T = 0$  the equality  $\sigma^T = 0$  holds. Hence and (30) we obtain the following

**Theorem 12.** *If the RPT-connection  $\nabla'$  on a Riemannian almost product  $\mathcal{W}_3$ -manifold  $(M, P, g)$  has a parallel torsion  $T$  and a Riemannian  $P$ -tensor of curvature  $R'$  then*

$$R(x, y, z, w) = R'(x, y, z, w) - \frac{1}{4}g(T(x, y), T(z, w)).$$

□

## 5. An Example

In the present section we study the RPT-connection  $\nabla'$  on an example of a Riemannian almost product  $\mathcal{W}_3$ -manifold constructed in [5] by a Lie group.

### 5.1. The Riemannian almost product $\mathcal{W}_3$ -manifold $(G, P, g)$

We will describe briefly the example of a Riemannian almost product  $\mathcal{W}_3$ -manifold in [5].

Let  $G$  be a 4-dimensional real connected Lie group and  $\mathfrak{g}$  be its Lie algebra with a basis  $\{X_i\}$ . By an introduction of a structure  $P$  and a left invariant metric  $g$  as follows

$$PX_1 = X_3, \quad PX_2 = X_4, \quad PX_3 = X_1, \quad PX_4 = X_2,$$

$$g(X_i, X_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \end{cases} \quad (34)$$

the manifold  $(G, P, g)$  becomes a Riemannian almost product manifold with  $\text{tr}P = 0$ .

The setting of the condition for the associated metric  $\tilde{g}$  to be a Killing metric [16], *i.e.*

$$g([X_i, X_j], PX_k) + g([X_i, X_k], PX_j) = 0, \quad (35)$$

specializes  $(G, P, g)$  as a Riemannian almost product  $\mathcal{W}_3$ -manifold and the Lie algebra  $\mathfrak{g}$  is determined by the following equalities:

$$\begin{aligned} [X_1, X_2] &= \lambda_1 X_1 + \lambda_2 X_2, & [X_1, X_3] &= \lambda_3 X_2 - \lambda_1 X_4, \\ [X_1, X_4] &= -\lambda_3 X_1 - \lambda_2 X_4, & [X_2, X_3] &= \lambda_4 X_2 + \lambda_1 X_3, \\ [X_2, X_4] &= -\lambda_4 X_1 + \lambda_2 X_3, & [X_3, X_4] &= \lambda_3 X_3 + \lambda_4 X_4, \end{aligned} \quad (36)$$

where  $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$ .

Further,  $(G, P, g)$  will stand for the Riemannian almost product  $\mathcal{W}_3$ -manifold determined by conditions (36).

In [5] there are computed the components with respect to the basis  $\{X_i\}$  of the tensor  $F$ , the Levi-Civita connection  $\nabla$ , the covariant derivative  $\nabla P$ , the curvature tensor  $R$  of  $\nabla$ .

There are computed also the scalar curvature  $\tau$  for  $\nabla$  and the square norm of  $\nabla P$ :

$$\tau = -\frac{5}{2} (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2), \quad \|\nabla P\|^2 = 4 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2). \quad (37)$$

### 5.2. The RPT-connection $\nabla'$ on $(G, P, g)$

Further in our considerations we exclude the trivial case  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$ , i.e. the case of a Riemannian  $P$ -manifold  $(G, P, g)$ .

From Proposition 7 and (37) we obtain the following

**Proposition 13.** *The manifold  $(G, P, g)$  has negative scalar curvatures with respect to the Levi-Civita connection  $\nabla$  and the RPT-connection  $\nabla'$ , as*

$$\tau' = -4 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2).$$

□

The following proposition is useful for computation of the components of the RPT-connection and its torsion.

**Proposition 14.** *The RPT-connection  $\nabla'$  on  $(G, P, g)$  and its torsion  $T$  satisfy the equalities:*

$$T(X_i, X_j) = -[PX_i, PX_j] \quad (38)$$

$$\nabla'_{X_i} X_j = [X_i, X_j] + P[X_i, PX_j]. \quad (39)$$

**Proof.** Because of (34) the following equality is valid

$$2g(\nabla_{X_i} X_j, X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j) + g([X_k, X_j], X_i). \quad (40)$$

Using (40) and (35), we obtain

$$2g(\nabla_{X_i} X_j, X_k) = g([X_i, X_j], X_k) + g(P[X_i, PX_j], X_k) - g(P[PX_i, X_j], X_k).$$

Then we have

$$2\nabla_{X_i} X_j = [X_i, X_j] + P[X_i, PX_j] - P[PX_i, X_j]. \quad (41)$$

The equality (41) implies

$$\begin{aligned} 2\nabla_{X_i} PX_j &= [X_i, PX_j] + P[X_i, X_j] - P[PX_i, PX_j], \\ 2P\nabla_{X_i} X_j &= P[X_i, X_j] + [X_i, PX_j] - [PX_i, X_j]. \end{aligned}$$

We subtract the last two equalities, apply the formula for the covariant derivation of  $P$  and obtain

$$2(\nabla_{X_i} P) X_j = -P[PX_i, PX_j] + [PX_i, X_j]. \quad (42)$$

Let  $\nabla'$  be the RPT-connection on  $(G, P, g)$ . By virtue of (11), (22) and (42), we have

$$\begin{aligned} 4T(X_i, X_j, X_k) = -g(3[PX_i, PX_j] - P[X_i, PX_j] \\ - P[PX_i, X_j] - [X_i, X_j], X_k). \end{aligned} \quad (43)$$

According to (36), we have

$$[PX_i, PX_j] + P[PX_i, X_j] + P[X_i, PX_j] + [X_i, X_j] = 0. \quad (44)$$

Bearing in mind (43) and (44), it follows (38).

From (8), (11), (41), (44) and (38), we obtain (39).  $\square$

Using (38) and (36), we obtain the following non-zero components  $T_{ijk} = T(X_i, X_j, X_k)$  of  $T$ :

$$T_{134} = -\lambda_1, \quad T_{234} = -\lambda_2, \quad T_{123} = -\lambda_3, \quad T_{124} = -\lambda_4. \quad (45)$$

The rest of the non-zero components are obtained from (45) by the totally skew-symmetry of  $T$ .

Combining (36) and (39), we get the components  $\nabla'_{X_i} X_j$  of the RPT-connection  $\nabla'$ :

$$\begin{aligned} \nabla'_{X_1} X_1 &= -\nabla'_{X_3} X_3 = -\lambda_1 X_2 + \lambda_3 X_4, \\ \nabla'_{X_2} X_2 &= -\nabla'_{X_4} X_4 = \lambda_2 X_1 - \lambda_4 X_3, \\ \nabla'_{X_1} X_2 &= -\nabla'_{X_3} X_4 = \lambda_1 X_1 - \lambda_3 X_3, \\ \nabla'_{X_1} X_3 &= -\nabla'_{X_3} X_1 = \lambda_3 X_2 - \lambda_1 X_4, \\ \nabla'_{X_1} X_4 &= -\nabla'_{X_3} X_2 = -\lambda_3 X_1 + \lambda_1 X_3, \\ \nabla'_{X_2} X_1 &= -\nabla'_{X_4} X_3 = -\lambda_2 X_2 + \lambda_4 X_4, \\ \nabla'_{X_2} X_3 &= -\nabla'_{X_4} X_1 = \lambda_4 X_2 - \lambda_2 X_4, \\ \nabla'_{X_2} X_4 &= -\nabla'_{X_4} X_2 = -\lambda_4 X_1 + \lambda_2 X_3. \end{aligned} \quad (46)$$

By virtue of (36) and (46), we get the following non-zero components  $R'_{ijks} = R'(X_i, X_j, X_k, X_s)$  of the tensor  $R'$ :

$$\begin{aligned} R'_{1212} = R'_{1234} &= \lambda_1^2 + \lambda_2^2, & R'_{1432} = R'_{1414} &= \lambda_2^2 + \lambda_3^2, \\ R'_{3412} = R'_{3434} &= \lambda_3^2 + \lambda_4^2, & R'_{2341} = R'_{2323} &= \lambda_1^2 + \lambda_4^2, \\ R'_{1314} = R'_{1332} = R'_{2441} &= R'_{2423} = \lambda_1 \lambda_2 - \lambda_3 \lambda_4, & & \\ R'_{1321} = R'_{1343} = R'_{2421} &= R'_{2443} = \lambda_1 \lambda_4 - \lambda_2 \lambda_3, & & \\ R'_{1241} = R'_{1223} = R'_{2312} = R'_{2334} &= R'_{1421}, & & \\ &= R'_{1443} = R'_{3441} = R'_{3423} = \lambda_1 \lambda_3 + \lambda_2 \lambda_4, & & \end{aligned} \quad (47)$$

The rest of the non-zero components are obtained from (47) by the properties  $R'_{ijks} = R'_{ksij}$  and  $R'_{ijks} = -R'_{jiks} = -R'_{ijsk}$ .

Using (45) and (46), we compute the following non-zero components  $(\nabla'_i T)_{jks} = (\nabla'_{X_i} T)(X_j, X_k, X_s)$  of  $\nabla' T$ :

$$\begin{aligned} (\nabla'_1 T)_{132} &= (\nabla'_2 T)_{142} = (\nabla'_3 T)_{134} = (\nabla'_4 T)_{234} = \lambda_1 \lambda_4 - \lambda_2 \lambda_3, \\ (\nabla'_1 T)_{143} &= (\nabla'_2 T)_{234} = (\nabla'_3 T)_{123} = (\nabla'_4 T)_{142} = \lambda_1 \lambda_2 - \lambda_3 \lambda_4, \\ (\nabla'_1 T)_{234} &= (\nabla'_3 T)_{142} = \lambda_1^2 - \lambda_3^2, \quad (\nabla'_2 T)_{143} = (\nabla'_4 T)_{123} = \lambda_2^2 - \lambda_4^2. \end{aligned} \quad (48)$$

The rest of the non-zero components are obtained from (48) by the totally skew-symmetry of  $T$ .

The exterior derivative  $dT$  of the totally skew-symmetric torsion  $T$  for the connection  $\nabla'$  on a Riemannian manifold  $(M, g)$  is given by the following formula (see e.g. [9])

$$dT(x, y, z, w) = \mathfrak{S}_{x,y,z} (\nabla'_x T)(y, z, w) - (\nabla'_w T)(x, y, z) + 2\sigma^T(x, y, z, w).$$

Then, using (45), (48) and (14), we compute that  $dT(X_i, X_j, X_k, X_s) = 0$  for all  $i, j, k, s$ . Therefore, the manifold  $(G, P, g)$  has a closed torsion 3-form  $T$  for the RPT-connection  $\nabla'$ . In this case we say that  $(G, P, g)$  has a *strong* RPT-structure by analogy with the Hermitian case. Therefore, we have the following

**Proposition 15.** *The manifold  $(G, P, g)$  has a strong RPT-structure.*  $\square$

In the following theorem we prove that the necessary and sufficient condition  $R'$  to be a Riemannian  $P$ -tensor and  $T$  to be parallel is one and the same.

**Theorem 16.** *Let  $\nabla'$  be the RPT-connection with curvature tensor  $R'$  and torsion  $T$  on the manifold  $(G, P, g)$ . Then the following conditions are equivalent:*

- (i)  $R'$  is a Riemannian  $P$ -tensor;
- (ii)  $T$  is parallel;
- (iii)  $\lambda_3 = \varepsilon\lambda_1$ ,  $\lambda_4 = \varepsilon\lambda_2$ ,  $\varepsilon = \pm 1$ .

**Proof.** As we have commented in Sec. 3, the RPT-connection  $\nabla'$  has a Riemannian  $P$ -tensor of curvature  $R'$  if and only if (5) is valid for  $R'$ . By virtue of (47) we establish directly that  $R'$  satisfies (5) if and only if conditions (iii) hold, i.e. (i) and (iii) are equivalent.

Using (48), we obtain immediately the equivalence of  $\nabla' T = 0$  and (iii), i.e. (ii) and (iii) are equivalent.  $\square$

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