A CLASSIFICATION OF THE TORSION TENSORS ON ALMOST CONTACT MANIFOLDS WITH B-METRIC

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Abstract. The space of the torsion (0,3)-tensors of all linear connections on almost contact manifolds with B-metric is decomposed in 11 orthogonal and invariant subspaces with respect to the action of the structural group. Thus, a classification of the connections on the considered manifolds with respect to the properties of their torsion tensor is generated. Two known natural connections are characterized regarding this classification.
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Introduction

The investigations of linear connections on almost contact manifolds with B-metric take a central place in the study of the differential geometry of these manifolds. Linear connections are characterized mainly by their torsion tensors. In accordance with our goals, it is important to classify the torsion tensors of linear connections regarding their properties with respect to the structures on the manifold.

Such a classification of the space of the torsion tensors is made in [3] in the case of almost complex manifolds with Norden metric. These manifolds are the even dimensional analogue of the odd dimensional almost contact manifolds.

The idea of decomposition of the space of the basic (0,3)-tensors generated by the covariant derivative of the fundamental tensor of type $(1,1)$ is used by different authors in order to obtain of the basic classifications of manifolds with additional tensor structures. For example, let us mention the classification of almost complex manifolds with Norden metric given in [1], of almost contact manifolds with B-metric – in [2], of Riemannian manifolds with trace-free almost product structure – in [11], of almost paracontact Riemannian manifolds of type (n,n) – in [8], of almost paracontact metric manifolds – in [10].

The goal of this work is to classify all torsion tensors with respect to the almost contact B-metric structure, which will be a base of further investigations of linear connections on these manifolds.

The present paper is organized as follows. In Sec. 1, we present some necessary facts about the considered manifolds. Section 2 is devoted to the decomposition of the space of torsion tensors on almost contact manifolds with B-metric. In Sec. 3, we find the position of two known natural connections in the obtained classification.

1. Almost Contact Manifolds with B-metric

Let (M,φ,ξ,η,g) be an almost contact manifold with B-metric or an *almost contact B-metric manifold*, i.e. M is a $(2n+1)$-dimensional differentiable manifold with an almost contact structure (φ,ξ,η) consisting of an endomorphism φ of the tangent bundle, a vector field ξ, its dual 1-form η as well as M is equipped with a pseudo-Riemannian metric g of signature $(n,n+1)$, such that the following algebraic relations are satisfied

$$\begin{gathered}\varphi\xi=0,\quad \varphi^2=-\operatorname{Id}+\eta\otimes\xi,\quad \eta\circ\varphi=0,\quad \eta(\xi)=1,\\ g(\varphi X,\varphi Y)=-g(X,Y)+\eta(X)\eta(Y)\end{gathered}\tag{1.1}$$

for arbitrary X, Y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M [2].

Further, X, Y, Z will stand for arbitrary elements of $\mathfrak{X}(M)$.

The structural group of (M,φ,ξ,η,g) is $G\times I$, where I is the identity on span(ξ) and $G=\mathcal{GL}(n;\mathbb{C})\cap\mathcal{O}(n,n)$. More precisely, it consists of real square matrices of order $2n+1$ of the following type

$$\left(\begin{array}{c|c|c} A & B & \vartheta^T\\ \hline -B & A & \vartheta^T\\ \hline \vartheta & \vartheta & 1\end{array}\right),\qquad \begin{array}{l} AA^T-BB^T=I_n,\\ AB^T+BA^T=O_n,\end{array}\qquad A,B\in\mathcal{GL}(n;\mathbb{R}),$$

where ϑ and its transpose ϑ^T are the zero row n-vector and the zero column n-vector; I_n and O_n are the unit matrix and the zero matrix of size n, respectively.

The associated metric $\tilde{g}$ of g on M is defined by

$$\tilde{g}(X,Y)=g(X,\varphi Y)+\eta(X)\eta(Y).$$

The manifold $(M,\varphi,\xi,\eta,\tilde{g})$ is also an almost contact B-metric manifold. Both metrics g and $\tilde{g}$ are necessarily of signature $(n,n+1)$. The Levi-Civita connection of g and $\tilde{g}$ will be denoted by ∇ and $\tilde{\nabla}$, respectively.

The Nijenhuis tensor N of the contact structure is defined by

$$N:=[\varphi,\varphi]+\mathrm{d}\eta\otimes\xi,$$

where $[\varphi,\varphi](X,Y)=[\varphi X,\varphi Y]+\varphi^2[X,Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]$ and $\mathrm{d}\eta$ is the exterior derivative of the 1-form η. By analogy with the skew-symmetric Lie bracket $[X,Y]=\nabla_XY-\nabla_YX$, let us consider the symmetric bracket $\{X,Y\}=\nabla_XY+\nabla_YX$. Hence we have $\{\varphi,\varphi\}(X,Y)=\varphi^2\{X,Y\}+\{\varphi X,\varphi Y\}-\varphi\{\varphi X,Y\}-\varphi\{X,\varphi Y\}$. Additionally, we use the Lie derivative with respect to ξ of the metric g, i.e. $(\mathcal{L}_\xi g)(X,Y)=(\nabla_X\eta)Y+(\nabla_Y\eta)X$, as an alternative of $\mathrm{d}\eta(X,Y) = (\nabla_X\eta)Y - (\nabla_Y\eta)X$. Then, we define the associated tensor S with N by:

$$S = \{\varphi,\varphi\} + (\mathcal{L}_\xi g)\otimes\xi. \tag{1.2}$$

A classification of almost contact manifolds with B-metric is given in [2]. This classification, consisting of eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{11}$, is made with respect to the tensor field F of type (0,3) defined by

$$F(X,Y,Z) = g\big((\nabla_X\varphi)Y, Z\big) \tag{1.3}$$

and having the following properties

$$\begin{aligned} F(X,Y,Z) = F(X,Z,Y) = F(X,\varphi Y,\varphi Z) &+ \eta(Y)F(X,\xi,Z)\\ &+ \eta(Z)F(X,Y,\xi). \end{aligned} \tag{1.4}$$

Further, x, y, z will stand for arbitrary vectors in the tangent space T_pM of M at an arbitrary point p in M. If $\{e_i\}$ $(i = 1,2,\dots,2n+1)$, as $e_{2n+1} = \xi$, is a basis of T_pM and $\left(g^{ij}\right)$ is the inverse matrix of (g_{ij}), then the following 1-forms are associated with F:

$$\begin{gathered} \theta(z) = g^{ij}F(e_i,e_j,z), \qquad \theta^*(z) = g^{ij}F(e_i,\varphi e_j,z),\\ \omega(z) = F(\xi,\xi,z). \end{gathered} \tag{1.5}$$

2. A Decomposition of the Space of Torsion Tensors

The object of our considerations are the linear connections with torsion. Thus, we have to study the properties of the torsion tensors with respect to the contact structure and the B-metric.

If T is the torsion tensor of D, i.e. $T(x,y) = D_xy - D_yx - [x,y]$, then the corresponding tensor of type (0,3) is determined by $T(x,y,z) = g(T(x,y),z)$.

Let us consider T_pM at arbitrary $p \in M$ as a $(2n+1)$-dimensional vector space with almost contact B-metric structure (V,φ,ξ,η,g). Moreover, let $\mathcal{T}$ be the vector space of all tensors T of type (0,3) over V having the skew-symmetry by the first two arguments, i.e.

$$\mathcal{T} = \{T(x,y,z) \in \mathbb{R} \mid \ T(x,y,z) = -T(y,x,z),\ x,y,z \in V\}.$$

The metric g induces an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{T}$ defined by

$$\langle T_1, T_2\rangle = g^{iq}g^{jr}g^{ks}T_1(e_i,e_j,e_k)T_2(e_q,e_r,e_s)$$

for arbitrary $T_1,\ T_2 \in \mathcal{T}$ and a basis $\{e_i\}$ $(i = 1,2,\dots,2n+1)$ of V.

The standard representation of the structural group $G\times I$ in V induces a natural representation λ of $G\times I$ in $\mathcal{T}$ as follows

$$((\lambda a)T)(x,y,z) = T\left(a^{-1}x, a^{-1}y, a^{-1}z\right)$$

for any $a \in G\times I$ and $T \in \mathcal{T}$, so that

$$\langle(\lambda a)T_1,(\lambda a)T_2\rangle = \langle T_1,T_2\rangle, \quad T_1,T_2 \in G\times I.$$

The decomposition $x = -\varphi^2 x + \eta(x)\xi$ generates the projectors h and v on V determined by $h(x) = -\varphi^2 x$ and $v(x) = \eta(x)\xi$ and having the properties

$$h \circ h = h, \quad v \circ v = v, \quad h \circ v = h \circ v = 0.$$

Therefore, we have the orthogonal decomposition $V = h(V) \oplus v(V)$.

Bearing in mind these projectors on V, we construct the partial decomposition of $\mathcal{T}$ as follows.

At first, we define the operator $p_1 : \mathcal{T} \to \mathcal{T}$ by

$$p_1(T)(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \quad T \in \mathcal{T}.$$

It is easy to check the following

Lemma 2.1. *The operator p_1 has the following properties:*

(i) $\langle p_1(T_1), T_2\rangle = \langle T_1, p_1(T_2)\rangle, \quad T_1, T_2 \in \mathcal{T}$;

(ii) $p_1 \circ p_1 = p_1$.

According to Lemma 2.1 we have

$$\mathcal{W}_1 = \mathrm{im}(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\},$$
$$\mathcal{W}_1^{\perp} = \ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.$$

Further, we consider the operator $p_2 : \mathcal{W}_1^{\perp} \to \mathcal{W}_1^{\perp}$ defined by

$$p_2(T)(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \quad T \in \mathcal{W}_1^{\perp}.$$

We obtain immediately the truthfulness of the following

Lemma 2.2. *The operator p_2 has the following properties:*

(i) $\langle p_2(T_1), T_2\rangle = \langle T_1, p_2(T_2)\rangle, \quad T_1, T_2 \in \mathcal{W}_1^{\perp}$;

(ii) $p_2 \circ p_2 = p_2$.

Then, bearing in mind Lemma 2.2, we obtain

$$\mathcal{W}_2 = \mathrm{im}(p_2) = \left\{T \in \mathcal{W}_1^{\perp} \mid p_2(T) = T\right\},$$
$$\mathcal{W}_2^{\perp} = \ker(p_2) = \left\{T \in \mathcal{W}_1^{\perp} \mid p_2(T) = 0\right\}.$$

Finally, we consider the operator $p_3 : \mathcal{W}_2^{\perp} \to \mathcal{W}_2^{\perp}$ defined by

$$p_3(T)(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z), \quad T \in \mathcal{W}_2^{\perp}$$

and we get the following

Lemma 2.3. *The operator p_3 has the following properties:*

(i) $\langle p_3(T_1), T_2\rangle = \langle T_1, p_3(T_2)\rangle, \quad T_1, T_2 \in \mathcal{W}_2^{\perp}$;

(ii) $p_3 \circ p_3 = p_3$.

By virtue of Lemma 2.3, we have

$$\mathcal{W}_3 = \mathrm{im}(p_3) = \left\{T \in \mathcal{W}_2^{\perp} \mid p_3(T) = T\right\},$$
$$\mathcal{W}_4 = \ker(p_3) = \left\{T \in \mathcal{W}_2^{\perp} \mid p_3(T) = 0\right\}.$$

From Lemma 2.1, Lemma 2.2 and Lemma 2.3 we have immediately

Theorem 2.4. *The decomposition*

$$\mathcal{T} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$$

is orthogonal and invariant under the action of the group $G \times I$. The subspaces $\mathcal{W}_i$ $(i = 1, 2, 3, 4)$ are determined by

$$\begin{aligned}
&\mathcal{W}_1: \quad T(x,y,z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z),\\
&\mathcal{W}_2: \quad T(x,y,z) = \eta(z) T(\varphi^2 x, \varphi^2 y, \xi),\\
&\mathcal{W}_3: \quad T(x,y,z) = \eta(x) T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y) T(\varphi^2 x, \xi, \varphi^2 z),\\
&\mathcal{W}_4: \quad T(x,y,z) = -\eta(z) \left\{ \eta(y) T(\varphi^2 x, \xi, \xi) + \eta(x) T(\xi, \varphi^2 y, \xi) \right\}
\end{aligned} \tag{2.1}$$

for arbitrary vectors $x, y, z \in V$.

Corollary 2.5. *The subspaces $\mathcal{W}_i$ $(i = 1, 2, 3, 4)$ are characterized as follows:*

$$\begin{aligned}
\mathcal{W}_1 &= \left\{ T \in \mathcal{T} \mid T(v(x), y, z) = T(x, y, v(z)) = 0 \right\},\\
\mathcal{W}_2 &= \left\{ T \in \mathcal{T} \mid T(v(x), y, z) = T(x, y, h(z)) = 0 \right\},\\
\mathcal{W}_3 &= \left\{ T \in \mathcal{T} \mid T(x, y, v(z)) = T(h(x), h(y), z) = 0 \right\},\\
\mathcal{W}_4 &= \left\{ T \in \mathcal{T} \mid T(x, y, h(z)) = T(h(x), h(y), z) = 0 \right\},
\end{aligned}$$

where $x, y, z \in V$.

The torsion forms associated with $T \in \mathcal{T}$ are defined as follows:

$$\begin{gathered}
t(x) = g^{ij} T(x, e_i, e_j), \quad t^*(x) = g^{ij} T(x, e_i, \varphi e_j),\\
\hat{t}(x) = T(x, \xi, \xi)
\end{gathered} \tag{2.2}$$

with respect to the basis $\{e_i; \xi\}$ $(i = 1, 2, \ldots, 2n)$ of V. Obviously, $\hat{t}(\xi) = 0$ is always valid.

According to Corollary 2.5, (2.1) and (2.2) we obtain the following

Corollary 2.6. *The torsion forms of T have the following properties in each of subspaces $\mathcal{W}_i$ $(i = 1, 2, 3, 4)$:*

(i) If $T \in \mathcal{W}_1$, then $t \circ v = t^* \circ v = \hat{t} = 0$;
(ii) If $T \in \mathcal{W}_2$, then $t = t^* = \hat{t} = 0$;
(iii) If $T \in \mathcal{W}_3$, then $t \circ h = t^* \circ h = \hat{t} = 0$;
(iv) If $T \in \mathcal{W}_4$, then $t = t^* = 0$.

Further we continue decomposition of the subspaces $\mathcal{W}_i$ $(i = 1, 2, 3, 4)$ of $\mathcal{T}$.

2.1. **The subspace $\mathcal{W}_1$.** Since the endomorphism φ induces an almost complex structure on the orthogonal complement $\{\xi\}^\perp$ of the subspace spanned by ξ and the restriction of g on $\{\xi\}^\perp$ is a Norden metric (because the almost complex structure causes an anti-isometry on $\{\xi\}^\perp$), then the decomposition of $\mathcal{W}_1$ is made as the decomposition of the space of the torsion tensors on an almost complex manifold with Norden metric known from [3].

Let us consider the linear operator $L_{10}: \mathcal{W}_1 \to \mathcal{W}_1$ defined by

$$L_{10}(T)(x,y,z) = -T(\varphi x, \varphi y, \varphi^2 z).$$

Then, it follows immediately

Lemma 2.7. *The operator L_{10} is an involutive isometry on $\mathcal{W}_1$ and commutes with the action of $G \times I$, i.e.*

$$L_{10} \circ L_{10} = \mathrm{id}_{\mathcal{W}_1}, \quad \langle L_{10}(T_1), L_{10}(T_2)\rangle = \langle T_1, T_2\rangle,$$
$$L_{10}((\lambda a)T) = (\lambda a)(L_{10}(T)),$$

where $T_1, T_2 \in \mathcal{T}$, $a \in G \times I$.

Therefore, the operator L_{10} has two eigenvalues $+1$ and -1, and the corresponding eigenspaces

$$\mathcal{W}_1^+ = \{T \in \mathcal{W}_1 \mid L_{10}(T) = T\}, \quad \mathcal{W}_1^- = \{T \in \mathcal{W}_1 \mid L_{10}(T) = -T\}$$

are invariant orthogonal subspaces of $\mathcal{W}_1$.

In order to decompose $\mathcal{W}_1^-$ we consider the linear operator $L_{11}: \mathcal{W}_1^- \to \mathcal{W}_1^-$ defined by

$$L_{11}(T)(x,y,z) = -T(\varphi x, \varphi^2 y, \varphi z).$$

We have

Lemma 2.8. *The operator L_{11} is an involutive isometry and commutes with the action of $G \times I$.*

According to the latter lemma, the eigenspaces

$$\mathcal{T}_{11} = \left\{T \in \mathcal{W}_1^- \mid L_{11}(T) = -T\right\}, \quad \mathcal{T}_{12} = \left\{T \in \mathcal{W}_1^- \mid L_{11}(T) = T\right\}$$

are invariant and orthogonal.

To decompose $\mathcal{W}_1^+$, we define the linear operator $L_{12}: \mathcal{W}_1^+ \to \mathcal{W}_1^+$ in the following way:

$$L_{12}(T)(x,y,z) = -\frac{1}{2}\left\{T(\varphi^2 y, \varphi^2 z, \varphi^2 x) + T(\varphi^2 z, \varphi^2 x, \varphi^2 y)\right.$$
$$\left. + T(\varphi y, \varphi^2 z, \varphi x) + T(\varphi^2 z, \varphi x, \varphi y)\right\}.$$

We obtain

Lemma 2.9. *The operator L_{12} is an involutive isometry and commutes with the action of $G \times I$.*

Thus, the eigenspaces

$$\mathcal{T}_{13} = \left\{T \in \mathcal{W}_1^+ \mid L_{12}(T) = -T\right\}, \quad \mathcal{T}_{14} = \left\{T \in \mathcal{W}_1^+ \mid L_{12}(T) = T\right\}$$

are invariant and orthogonal.

Using Lemma 2.7, Lemma 2.8 and Lemma 2.9, we get the following

Theorem 2.10. *The decomposition*

$$\mathcal{W}_1 = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{13} \oplus \mathcal{T}_{14}$$

is orthogonal and invariant with respect to the structural group.

Bearing in mind the definition of the subspaces $\mathcal{T}_{1i}$ $(i = 1, 2, 3, 4)$, we obtain

Proposition 2.11. *The subspaces of* $\mathcal{W}_1$ *are determined by:*

$$\begin{aligned}
\mathcal{T}_{11}: \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
& T(x, y, z) = -T(\varphi x, \varphi y, z) = -T(x, \varphi y, \varphi z); \\
\Leftrightarrow \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
& T(\varphi x, y, z) = T(x, \varphi y, z) = T(x, y, \varphi z); \\
\mathcal{T}_{12}: \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
& T(x, y, z) = -T(\varphi x, \varphi y, z) = T(\varphi x, y, \varphi z); \\
\mathcal{T}_{13}: \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
& T(x, y, z) - T(\varphi x, \varphi y, z) = \underset{x,y,z}{\mathfrak{S}} T(x, y, z) = 0; \\
\mathcal{T}_{14}: \quad & T(\xi, y, z) = T(x, y, \xi) = 0, \\
& T(x, y, z) - T(\varphi x, \varphi y, z) = \underset{x,y,z}{\mathfrak{S}} T(\varphi x, y, z) = 0.
\end{aligned}$$

Using Corollary 2.6 (i), Proposition 2.11 and (2.2), we obtain

Corollary 2.12. *The torsion forms* t *and* t^* *of* T *have the following properties in the subspaces* $\mathcal{T}_{1i}$ $(i = 1, 2, 3, 4)$ *of* $\mathcal{W}_1$*:*

(i) *If* $T \in \mathcal{T}_{11}$, *then* $t(x) = -t^*(\varphi x)$, $t(\varphi x) = t^*(x)$;
(ii) *If* $T \in \mathcal{T}_{12}$, *then* $t = t^* = 0$;
(iii) *If* $T \in \mathcal{T}_{13}$, *then* $t(x) = t^*(\varphi x)$, $t(\varphi x) = -t^*(x)$.
(iv) *If* $T \in \mathcal{T}_{14}$, *then* $t = t^* = 0$;

Let us remark that each of the subspaces $\mathcal{T}_{11}$ and $\mathcal{T}_{13}$ can be decompose additionally to a couple of subspaces — one of zero traces $(t,\ t^*)$ and one of non-zero traces $(t,\ t^*)$.

The projection operators of $\mathcal{T}$ in $\mathcal{T}_{1i}$ $(i = 1, 2, 3, 4)$, following [3], are given in the next

Proposition 2.13. *Let $T \in \mathcal{T}$ and p_{1i} $(i = 1, 2, 3, 4)$ be the projection operators of $\mathcal{T}$ in $\mathcal{T}_{1i}$. Then*

$$\begin{aligned} p_{11}(T)(x,y,z) = -\frac{1}{4}\big\{&T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z) \\ &-T(\varphi x, \varphi^2 y, \varphi z) - T(\varphi^2 x, \varphi y, \varphi z)\big\}; \\ p_{12}(T)(x,y,z) = -\frac{1}{4}\big\{&T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z) \\ &+T(\varphi x, \varphi^2 y, \varphi z) + T(\varphi^2 x, \varphi y, \varphi z)\big\}; \\ p_{13}(T)(x,y,z) = -\frac{1}{8}\big\{&2T(\varphi^2 x, \varphi^2 y, \varphi^2 z) + 2T(\varphi x, \varphi y, \varphi^2 z) \\ &- T(\varphi^2 y, \varphi^2 z, \varphi^2 x) - T(\varphi^2 z, \varphi^2 x, \varphi^2 y) \\ &-T(\varphi y, \varphi^2 z, \varphi x) - T(\varphi^2 z, \varphi x, \varphi y) \\ &- T(\varphi y, \varphi z, \varphi^2 x) - T(\varphi z, \varphi x, \varphi^2 y) \\ &+T(\varphi^2 y, \varphi z, \varphi x) + T(\varphi z, \varphi^2 x, \varphi y)\big\}; \\ p_{14}(T)(x,y,z) = -\frac{1}{8}\big\{&2T(\varphi^2 x, \varphi^2 y, \varphi^2 z) + 2T(\varphi x, \varphi y, \varphi^2 z) \\ &+ T(\varphi^2 y, \varphi^2 z, \varphi^2 x) + T(\varphi^2 z, \varphi^2 x, \varphi^2 y) \\ &+T(\varphi y, \varphi^2 z, \varphi x) + T(\varphi^2 z, \varphi x, \varphi y) \\ &+T(\varphi y, \varphi z, \varphi^2 x) + T(\varphi z, \varphi x, \varphi^2 y) \\ &-T(\varphi^2 y, \varphi z, \varphi x) - T(\varphi z, \varphi^2 x, \varphi y)\big\}. \end{aligned}$$

Proof. Let us show the calculations about p_{11} for example. Lemma 2.7 implies that the tensor $\frac{1}{2}\{T - L_{10}(T)\}$ is the projection of T in $\mathcal{W}_1^- = \mathcal{T}_{11} \oplus \mathcal{T}_{12}$. Using Lemma 2.8, we find the expression of p_{11} in terms of the operators L_{10} and L_{11}, i.e.

$$p_{11}(T) = \frac{1}{4}\{T - L_{10}(T) - L_{11}(T) + L_{11} \circ L_{10}(T)\},$$

which implies the stated expression of p_{11}. In a similar way we prove the expressions for the other projectors under consideration.

We verify that $p_{1i} \circ p_{1i} = p_{1i}$ and $\sum_i p_{1i} = \mathrm{id}_{\mathcal{W}_1}$ for $i = 1, 2, 3, 4$. □

2.2. **The subspace $\mathcal{W}_2$.** Following the demonstrated procedure for $\mathcal{W}_1$, we continue the decomposition of the other main subspaces of $\mathcal{T}$ with respect to the almost contact B-metric structure.

Lemma 2.14. *The operator L_{20}, defined by*

$$L_{20}(T)(x,y,z) = \eta(z)T(\varphi x, \varphi y, \xi),$$

is an involutive isometry on $\mathcal{W}_2$ *and commutes with the action of the group* $G \times I$.

Hence, the corresponding eigenspaces, determined by

$$\mathcal{T}_{21} = \{T \in \mathcal{W}_2 \mid L_{20}(T) = -T\}, \quad \mathcal{T}_{22} = \{T \in \mathcal{W}_2 \mid L_{20}(T) = T\},$$

are invariant and orthogonal. Therefore, we have

Theorem 2.15. *The decomposition*

$$\mathcal{W}_2 = \mathcal{T}_{21} \oplus \mathcal{T}_{22}$$

is orthogonal and invariant with respect to the structural group.

Proposition 2.16. *The subspaces of* $\mathcal{W}_2$ *are determined by:*

$$\mathcal{T}_{21}: \quad T(x,y,z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \quad T(x,y,\xi) = -T(\varphi x, \varphi y, \xi);$$

$$\mathcal{T}_{22}: \quad T(x,y,z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \quad T(x,y,\xi) = T(\varphi x, \varphi y, \xi).$$

Then the tensors $\frac{1}{2}\{T - L_{20}(T)\}$ and $\frac{1}{2}\{T + L_{20}(T)\}$ are the projections of $\mathcal{W}_2$ in $\mathcal{T}_{21}$ and $\mathcal{T}_{22}$, respectively. Moreover, we have $p_{2j} \circ p_{2j} = p_{2j}$ $(j = 1, 2)$ and $p_{21} + p_{22} = \mathrm{id}_{\mathcal{W}_2}$. Therefore we obtain

Proposition 2.17. *Let* $T \in \mathcal{T}$ *and* p_{2j} $(j = 1, 2)$ *be the projection operators of* $\mathcal{T}$ *in* $\mathcal{T}_{2j}$. *Then*

$$p_{21}(T)(x,y,z) = \frac{1}{2}\eta(z)\left\{T(\varphi^2 x, \varphi^2 y, \xi) - T(\varphi x, \varphi y, \xi)\right\},$$

$$p_{22}(T)(x,y,z) = \frac{1}{2}\eta(z)\left\{T(\varphi^2 x, \varphi^2 y, \xi) + T(\varphi x, \varphi y, \xi)\right\}.$$

According to Proposition 2.6 (ii), Proposition 2.16 and (2.2) we obtain the following

Corollary 2.18. *The torsion forms of* T *are zero in each of subspaces* $\mathcal{T}_{21}$ *and* $\mathcal{T}_{22}$, *i.e. if* $T \in \mathcal{T}_{2j}$ $(j = 1, 2)$, *then* $t = t^* = \hat{t} = 0$.

2.3. **The subspace** $\mathcal{W}_3$**.**

Lemma 2.19. *The following operators* L_{3k} $(k = 0, 1)$ *are involutive isometries on* $\mathcal{W}_3$ *and commute with the action of the group* $G \times I$:

$$L_{30}(T)(x,y,z) = \eta(x)T(\xi, \varphi y, \varphi z) - \eta(y)T(\xi, \varphi x, \varphi z),$$

$$L_{31}(T)(x,y,z) = \eta(x)T(\xi, \varphi^2 z, \varphi^2 y) - \eta(y)T(\xi, \varphi^2 z, \varphi^2 x).$$

By virtue of their action, we obtain consecutively the corresponding invariant and orthogonal eigenspaces:

$$\mathcal{W}_3^- = \{T \in \mathcal{W}_3 \mid L_{30}(T) = -T\}, \quad \mathcal{W}_3^+ = \{T \in \mathcal{W}_3 \mid L_{30}(T) = T\},$$
$$\mathcal{T}_{31} = \left\{T \in \mathcal{W}_3^- \mid L_{31}(T) = T\right\}, \quad \mathcal{T}_{32} = \left\{T \in \mathcal{W}_3^- \mid L_{31}(T) = -T\right\},$$
$$\mathcal{T}_{33} = \left\{T \in \mathcal{W}_3^+ \mid L_{31}(T) = T\right\}, \quad \mathcal{T}_{34} = \left\{T \in \mathcal{W}_3^+ \mid L_{31}(T) = -T\right\}.$$

In such a way, we get

Theorem 2.20. *The decomposition*

$$\mathcal{W}_3 = \mathcal{T}_{31} \oplus \mathcal{T}_{32} \oplus \mathcal{T}_{33} \oplus \mathcal{T}_{34}$$

is orthogonal and invariant with respect to the structural group.

Proposition 2.21. *The subspaces of* $\mathcal{W}_3$ *are determined by:*

$$\begin{aligned}
\mathcal{T}_{31}: \quad & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
& T(\xi,y,z) = T(\xi,z,y) = -T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{32}: \quad & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
& T(\xi,y,z) = -T(\xi,z,y) = -T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{33}: \quad & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
& T(\xi,y,z) = T(\xi,z,y) = T(\xi,\varphi y,\varphi z);\\
\mathcal{T}_{34}: \quad & T(x,y,z) = \eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z),\\
& T(\xi,y,z) = -T(\xi,z,y) = T(\xi,\varphi y,\varphi z).
\end{aligned}$$

By virtue of Corollary 2.6 (iii), Proposition 2.21 and (2.2) we obtain

Corollary 2.22. *The torsion forms* t *and* t^* *of* T*:*

(i) are zero in the subspaces $\mathcal{T}_{3k}$ $(k = 2,3,4)$ *of* $\mathcal{W}_3$*;*
(ii) have no extra properties in $\mathcal{T}_{31} \subset \mathcal{W}_3$*.*

Let us remark that $\mathcal{T}_{31}$ can be decompose additionally to three subspaces determined by conditions $t = 0$, $t^* = 0$ and $t = t^* = 0$, respectively.

Proposition 2.23. *Let $T \in \mathcal{T}$ and p_{3k} $(k = 1, 2, 3, 4)$ be the projection operators of $\mathcal{T}$ in $\mathcal{T}_{3k}$. Then*

$$\begin{aligned}
p_{31}(T)(x,y,z) =& \frac{1}{4}\big\{\eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z) \\
&+ \eta(x)T(\xi,\varphi^2 z,\varphi^2 y) - \eta(y)T(\xi,\varphi^2 z,\varphi^2 x) \\
&- \eta(x)T(\xi,\varphi y,\varphi z) + \eta(y)T(\xi,\varphi x,\varphi z) \\
&- \eta(x)T(\xi,\varphi z,\varphi y) + \eta(y)T(\xi,\varphi z,\varphi x)\big\};\\
p_{32}(T)(x,y,z) =& \frac{1}{4}\big\{\eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z) \\
&- \eta(x)T(\xi,\varphi^2 z,\varphi^2 y) + \eta(y)T(\xi,\varphi^2 z,\varphi^2 x) \\
&- \eta(x)T(\xi,\varphi y,\varphi z) + \eta(y)T(\xi,\varphi x,\varphi z) \\
&+ \eta(x)T(\xi,\varphi z,\varphi y) - \eta(y)T(\xi,\varphi z,\varphi x)\big\};\\
p_{33}(T)(x,y,z) =& \frac{1}{4}\big\{\eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z) \\
&+ \eta(x)T(\xi,\varphi^2 z,\varphi^2 y) - \eta(y)T(\xi,\varphi^2 z,\varphi^2 x) \\
&+ \eta(x)T(\xi,\varphi y,\varphi z) - \eta(y)T(\xi,\varphi x,\varphi z) \\
&+ \eta(x)T(\xi,\varphi z,\varphi y) - \eta(y)T(\xi,\varphi z,\varphi x)\big\};\\
p_{34}(T)(x,y,z) =& \frac{1}{4}\big\{\eta(x)T(\xi,\varphi^2 y,\varphi^2 z) - \eta(y)T(\xi,\varphi^2 x,\varphi^2 z) \\
&- \eta(x)T(\xi,\varphi^2 z,\varphi^2 y) + \eta(y)T(\xi,\varphi^2 z,\varphi^2 x) \\
&+ \eta(x)T(\xi,\varphi y,\varphi z) - \eta(y)T(\xi,\varphi x,\varphi z) \\
&- \eta(x)T(\xi,\varphi z,\varphi y) + \eta(y)T(\xi,\varphi z,\varphi x)\big\}.
\end{aligned}$$

2.4. **The subspace $\mathcal{W}_4$.** Finally, we denote only $\mathcal{W}_4$ as $\mathcal{T}_{41}$ and it is determined as follows

$$\mathcal{T}_{41}: \quad T(x,y,z) = \eta(z)\left\{\eta(y)\hat{t}(x) - \eta(x)\hat{t}(y)\right\}. \tag{2.3}$$

Obviously, the projection operator $p_{41} : \mathcal{T} \to \mathcal{T}_{41}$ has the form

$$p_{41}(T)(x,y,z) = \eta(z)\left\{\eta(y)\hat{t}(x) - \eta(x)\hat{t}(y)\right\}. \tag{2.4}$$

As conclusion, combining Theorem 2.4, Theorem 2.10, Theorem 2.15 and Theorem 2.20, we obtain the following main statement in this paper.

Theorem 2.24. *The following decomposition in 11 factors of the vector space $\mathcal{T}$ of the torsion tensors of type (0,3) of $(M, \varphi, \xi, \eta, g)$ is orthogonal and invariant with respect to the structural group $G \times I$:*

$$\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{13} \oplus \mathcal{T}_{14} \oplus \mathcal{T}_{21} \oplus \mathcal{T}_{22} \oplus \mathcal{T}_{31} \oplus \mathcal{T}_{32} \oplus \mathcal{T}_{33} \oplus \mathcal{T}_{34} \oplus \mathcal{T}_{41}.$$

3. The Classes of the Torsion Tensors and Known Natural Connections

Let (M,φ,ξ,η,g) be an almost contact B-metric manifold. The tangent space T_pM at an arbitrary point p in M is a vector space equipped with an almost contact B-metric structure.

There exist a countless number of linear connections on the tangent bundle TM and each of them has a torsion tensor T as its fundamental invariant. Then the subspace $\mathcal{T}_{ij}$, where T belongs, is an important characteristic of the connection D. This statement is based on the following relation [4]:

$$g\left(D_XY-\nabla_XY,Z\right)=\frac{1}{2}\left\{T(X,Y,Z)-T(Y,Z,X)+T(Z,X,Y)\right\}.$$

In such a way the conditions for T described as the subspace $\mathcal{T}_{ij}$ give rise to the corresponding class of the connection with respect to its torsion tensor.

Thus, the conditions define the eleven basic classes of the torsion tensors of the connections on the tangent bundle generated by an almost contact B-metric manifold. Of course, the number of all classes under conversation is 2^{11} and their defining conditions are easily obtainable by the basic ones.

The special class $\mathcal{T}_{00}$ of the symmetric connections is defined by the condition $T=0$. This class belongs to each of the defined classes. The Levi-Civita connections ∇ and $\tilde{\nabla}$ are symmetric and therefore they belongs to the class $\mathcal{T}_{00}$.

In the following two subsections we discuss about two known natural connections with torsion on (M,φ,ξ,η,g). Natural connections are a generalization of Levi-Civita connections. A linear connection is called a *natural connection* on (M,φ,ξ,η,g) if the almost contact structure (φ,ξ,η) and the B-metric g (consequently also $\tilde{g}$) are parallel with respect to it. Let us recall the following

Proposition 3.1 ([9]). *A linear connection D is a natural connection on (M,φ,ξ,η,g) if and only if*

$$Q(x,y,\varphi z)-Q(x,\varphi y,z)=F(x,y,z), \tag{3.1}$$

$$Q(x,y,z)=-Q(x,z,y), \tag{3.2}$$

where $Q(x,y,z)=g\left(D_xy-\nabla_xy,z\right)$.

Now, we will prove the following

Theorem 3.2. *A linear connection D is natural on (M,φ,ξ,η,g) if and only if $D\varphi=Dg=0$.*

Proof. According to the proof of Proposition 3.1 in [9], conditions (3.1) and (3.2) are equivalent to $D\varphi=0$ and $Dg=0$, respectively. Moreover, $D\xi=0$ is equivalent to the relation $Q(x,\xi,z)=-F(x,\xi,\varphi z)$, which is a consequence of (3.1). Finally, since $\eta(\cdot)=g(\cdot,\xi)$, then supposing $Dg=0$ we have $D\xi=0$ if and only if $D\eta=0$. Thus, the statement is truthful. □

Let us remark that in [12] there are introduced families of linear connections with different properties with respect to the almost contact B-metric structure, which can be described in the frame of present classification regarding the torsion tensor.

3.1. **The canonical connection in the classification of the torsion tensors.** In [7] is introduced a natural connection D on (M,φ,ξ,η,g) by

$$D_x y = \nabla_x y + \frac{1}{2}\big\{(\nabla_x\varphi)\,\varphi y + (\nabla_x\eta)\,y\cdot\xi\big\} - \eta(y)\nabla_x\xi.$$

This connection, studied for some of the basic classes in [5] and [6] under the name a *canonical connection*, has a torsion tensor and the torsion forms as follows:

$$\begin{aligned} T(x,y,z) = &-\frac{1}{2}\big\{F(x,\varphi y,\varphi^2 z) - F(y,\varphi x,\varphi^2 z)\big\}\\ &+\eta(z)\big\{F(x,\varphi y,\xi) - F(y,\varphi x,\xi)\big\}\\ &+\eta(x)F(y,\varphi z,\xi) - \eta(y)F(x,\varphi z,\xi), \end{aligned} \tag{3.3}$$

$$t(x) = \frac{1}{2}\left\{\theta^*(x) + \theta^*(\xi)\eta(x)\right\}, \quad t^*(x) = -\frac{1}{2}\left\{\theta(x) + \theta(\xi)\eta(x)\right\},$$
$$\hat{t}(x) = -\omega(\varphi x).$$

We establish the position of D in the classification above by the following

Theorem 3.3. *The canonical connection belongs to the following subspace of* $\mathcal{T}$*:*

$$\mathcal{T}_{12}\oplus\mathcal{T}_{13}\oplus\mathcal{T}_{14}\oplus\mathcal{T}_{21}\oplus\mathcal{T}_{22}\oplus\mathcal{T}_{31}\oplus\mathcal{T}_{32}\oplus\mathcal{T}_{33}\oplus\mathcal{T}_{34}\oplus\mathcal{T}_{41}.$$

Proof. Applying Proposition 2.13, Proposition 2.17, Proposition 2.23 and (2.4) for the torsion tensor T from (3.3), we obtain the components of T in each of subspaces $\mathcal{T}_{ij}$:

$$p_{11}(T)(x,y,z) = 0,$$

$$\begin{aligned} p_{12}(T)(x,y,z) = -\frac{1}{4}\big\{&F(\varphi^2 x,\varphi^2 y,\varphi z) - F(\varphi^2 y,\varphi^2 x,\varphi z)\\ &-F(\varphi x,\varphi y,\varphi z) + F(\varphi y,\varphi x,\varphi z)\big\}, \end{aligned}$$

$$\begin{aligned} p_{13}(T)(x,y,z) = -\frac{1}{8}\big\{&2F(\varphi^2 z,\varphi^2 y,\varphi x) + F(\varphi^2 x,\varphi^2 y,\varphi z)\\ &-F(\varphi^2 y,\varphi^2 x,\varphi z) + F(\varphi x,\varphi y,\varphi z) - F(\varphi y,\varphi x,\varphi z)\big\}, \end{aligned}$$

$$\begin{aligned} p_{14}(T)(x,y,z) = \frac{1}{8}\big\{&2F(\varphi^2 z,\varphi^2 y,\varphi x) - F(\varphi^2 x,\varphi^2 y,\varphi z)\\ &+F(\varphi^2 y,\varphi^2 x,\varphi z) - F(\varphi x,\varphi y,\varphi z) + F(\varphi y,\varphi x,\varphi z)\big\}, \end{aligned}$$

$$
\begin{aligned}
p_{21}(T)(x,y,z) = -\frac{1}{2}\eta(z)\left\{F(\varphi^2 x,\varphi y,\xi) - F(\varphi^2 y,\varphi x,\xi)\right. \\
\left. -F(\varphi x,y,\xi) + F(\varphi y,x,\xi)\right\}, \\
p_{22}(T)(x,y,z) = -\frac{1}{2}\eta(z)\left\{F(\varphi^2 x,\varphi y,\xi) - F(\varphi^2 y,\varphi x,\xi)\right. \\
\left. +F(\varphi x,y,\xi) - F(\varphi y,x,\xi)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&p_{31}(T)(x,y,z) = \\
&= -\frac{1}{4}\eta(x)\big\{F(\varphi^2 y,\varphi z,\xi) + F(\varphi^2 z,\varphi y,\xi) - F(\varphi y,z,\xi) - F(\varphi z,y,\xi)\big\} \\
&\quad + \frac{1}{4}\eta(y)\big\{F(\varphi^2 x,\varphi z,\xi) + F(\varphi^2 z,\varphi x,\xi) - F(\varphi x,z,\xi) - F(\varphi z,x,\xi)\big\}, \\
&p_{32}(T)(x,y,z) = \\
&= -\frac{1}{4}\eta(x)\big\{F(\varphi^2 y,\varphi z,\xi) - F(\varphi^2 z,\varphi y,\xi) - F(\varphi y,z,\xi) + F(\varphi z,y,\xi)\big\} \\
&\quad + \frac{1}{4}\eta(y)\big\{F(\varphi^2 x,\varphi z,\xi) - F(\varphi^2 z,\varphi x,\xi) - F(\varphi x,z,\xi) + F(\varphi z,x,\xi)\big\}, \\
&p_{33}(T)(x,y,z) = \\
&= -\frac{1}{4}\eta(x)\big\{F(\varphi^2 y,\varphi z,\xi) + F(\varphi^2 z,\varphi y,\xi) + F(\varphi y,z,\xi) + F(\varphi z,y,\xi)\big\} \\
&\quad + \frac{1}{4}\eta(y)\big\{F(\varphi^2 x,\varphi z,\xi) + F(\varphi^2 z,\varphi x,\xi) + F(\varphi x,z,\xi) + F(\varphi z,x,\xi)\big\}, \\
&p_{34}(T)(x,y,z) = \\
&= -\frac{1}{4}\eta(x)\big\{F(\varphi^2 y,\varphi z,\xi) - F(\varphi^2 z,\varphi y,\xi) + F(\varphi y,z,\xi) - F(\varphi z,y,\xi) \\
&\qquad\qquad + 2F(\xi,\varphi y,\varphi^2 z)\big\} \\
&\quad + \frac{1}{4}\eta(y)\big\{F(\varphi^2 x,\varphi z,\xi) - F(\varphi^2 z,\varphi x,\xi) + F(\varphi x,z,\xi) - F(\varphi z,x,\xi) \\
&\qquad\qquad + 2F(\xi,\varphi x,\varphi^2 z)\big\},
\end{aligned}
$$

$$
p_{41}(T)(x,y,z) = \eta(z)\left\{\eta(x)\omega(\varphi y) - \eta(y)\omega(\varphi x)\right\}.
$$

□

3.2. **The φKT-connection in the classification of the torsion tensors.** In [9], it is introduced a natural connection D on (M,φ,ξ,η,g), called a *φKT-connection*, which torsion tensor T is totally skew-symmetric, i.e. a 3-form. It is proved that this connection exists only on the class $\mathcal{F}_3 \oplus \mathcal{F}_7$, i.e.

the class of almost contact B-metric manifolds, where ξ is a Killing vector field and the cyclic sum $\mathfrak{S}$ of F by three arguments is zero. Alternatively, the class $\mathcal{F}_3 \oplus \mathcal{F}_7$ is characterized by the condition $S = 0$, defined by (1.2). Then, D is determined by

$$g(D_x y, z) = g(\nabla_x y, z) + \frac{1}{2}T(x, y, z),$$

where the torsion tensor T is defined by

$$T(x, y, z) = -\frac{1}{2} \mathop{\mathfrak{S}}_{x,y,z} \big\{F(x, y, \varphi z) - 3\eta(x)F(y, \varphi z, \xi)\big\}. \tag{3.4}$$

Obviously, the torsion forms of the φKT-connection are zero.

From (3.4), in a similar way of Theorem 3.3, we get the following non-zero components of T:

$$\begin{aligned} p_{12}(T)(x, y, z) &= -\frac{1}{2}\big\{F(x, y, \varphi z) + F(y, z, \varphi x) - F(z, x, \varphi y) \\ &\qquad + \eta(y)F(z, \varphi x, \xi) + \eta(z)F(x, \varphi y, \xi)\big\}, \\ p_{14}(T)(x, y, z) &= -F(z, x, \varphi y) - \frac{1}{2}\eta(x)F(y, \varphi z, \xi), \\ p_{21}(T)(x, y, z) &= 2\eta(z)F(x, \varphi y, \xi), \\ p_{32}(T)(x, y, z) &= 2\eta(x)F(y, \varphi z, \xi) + 2\eta(y)F(z, \varphi x, \xi). \end{aligned}$$

Therefore we have

Theorem 3.4. *The torsion tensor of the φKT-connection belongs to the following subspace of $\mathcal{T}$:*

$$\mathcal{T}_{12} \oplus \mathcal{T}_{14} \oplus \mathcal{T}_{21} \oplus \mathcal{T}_{32}.$$

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