ON THE STRUCTURE OF THE FINITE-DIMENSIONAL COMMUTATIVE SEMISIMPLE ALGEBRAS

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ВЪРХУ СТРУКТУРАТА НА КРАЙНОМЕРНИТЕ КОМУТАТИВНИ ПОЛУПРОСТИ АЛГЕБРИ

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Резюме

В статията се извежда критерий кога една крайномерна комутативна полупроста алгебра над алгебрично затворено поле F е изоморфна като F-алгебра на групова алгебра FG на крайна абелева група G. Така ние даваме частично решение на Проблем 1 на Brauer. Изследва се структурата на крайномерните комутативни полупрости алгебри над полето $\mathbf{R}$ на реалните числа. Освен това се извежда необходимо и достатъчно условие една крайномерна комутативна алгебра над полето $\mathbf{R}$ да е изоморфна като $\mathbf{R}$-алгебра на някоя реална групова алгебра.

***Ключови думи**: крайномерна комутативна алгебра; групова алгебра; изоморфизъм на алгебри; реална мощност на алгебра*

1. Introduction

In the present paper we examine the structure of the finite-dimensional commutative semisimple algebras over an algebraically closed field and over the field $\mathbf{R}$. We give a criterion for a finite-dimensional commutative semisimple algebra over an algebraically closed field F to be isomorphic as an F-algebra to a group algebra FG of a finite abelian group G. Thus, we give a partial solution to Brauer's Problem 1 (Brauer 1963). We consider the structure of real finite-dimensional commutative semisimple algebras and we describe it up to isomorphism. We define the concept real cardinality of a commutative semisimple algebra over $\mathbf{R}$ and we give a necessary and sufficient condition for such algebra to be isomorphic as an $\mathbf{R}$-algebra to a group algebra $\mathbf{R}G$ of a finite abelian group G. Moreover, we find a necessary and sufficient condition for a finite-dimensional commutative algebra over $\mathbf{R}$ to be isomorphic as an $\mathbf{R}$-algebra to some real group algebra.

If G is a finite multiplicative abelian group then we denote $G[2]=\{g\in G \mid g^2=1\}$ in the whole paper.

2. Structure of a finite-dimensional commutative semisimple algebras over an algebraically closed field

In the theory of group algebras the following fact which is a partial case of the result of (May 1971) is well known:

If G and $\overline{G}$ are torsion abelian groups and F is an algebraically closed field of characteristic 0, then the group algebras FG and $F\overline{G}$ are isomorphic as F-algebras if and *only if* $|G| = |\overline{G}|$.

We prove the following result:

Proposition. *Let* F *be an algebraically closed field and* A *be a commutative semisimple algebra over* F *with* $\dim_F A = n$ *(*$n \in N$*). Then* A *is isomorphic as an* F*-algebra to the group algebra* FG *of the abelian group* G *of order* n*.*

Proof. For the finite-dimensional commutative semisimple F-algebra A we apply the structural theorem of Wedderburn (Gluhov, Elizarov, Nechaev 2003, Pierce 1986, Lam 2001) and we get

$$A \cong M_{n_1}(F) \oplus M_{n_2}(F) \oplus ... \oplus M_{n_s}(F),$$

where $n_1^2 + n_2^2 + ... + n_s^2 = n$. Since A is a commutative algebra, then $M_{n_i}(F)$ is a commutative algebra for each $i = 1,2,...,s$. Therefore $n_i = 1$ for $i = 1,2,...,s$, which leads to

$$A \cong F \oplus F \oplus ... \oplus F,$$

where the number of the direct addends is n.

On the other hand, according to (Passman 2011), if G is an abelian group of order n, then

$$FG \cong F \oplus F \oplus ... \oplus F,$$

where the number of direct addends is equal to the order of the group G. Therefore A is isomorphic to the group algebra FG as an F-algebra.

Using this proposition in the case when F is the field $\mathbf{C}$ of the complex numbers we give a partial solution to the following Brauer's Problem 1 (Brauer 1963): what are the possible complex group algebras of finite groups?

3. Structure of real finite-dimensional commutative semisimple algebras

There are a number of researches of the infinite-dimensional commutative semisimple algebras over the field $\mathbf{R}$ of the real numbers. Important results for real group algebras are obtained by Berman (Berman 1967) who finds a full system of invariants of a group algebra of infinitely countable torsion abelian group over the field $\mathbf{R}$. Berman and Bogdan (Berman, Bogdan 1977) generalise this result for arbitrary infinite abelian groups. The normed multiplicative group of a real group algebra of an abelian p-group is described by Mollov (Mollov 1984).

In this section we will examine the structure of the finite-dimensional commutative semisimple algebras over the field of the real numbers.

Theorem 1. *Let* A *be a real finite-dimensional commutative semisimple algebra. Then*

$$A \cong \mathbf{R} \oplus ... \oplus \mathbf{R} \oplus \mathbf{C} \oplus ... \oplus \mathbf{C}. \quad (1)$$

Proof. Let $\dim_{\mathbf{R}} A = n$ ($n \in N$). According to the structural theorem of Wedderburn (Gluhov, Elizarov, Nechaev 2003, Pierce 1986, Lam 2001) applied to the semisimple algebra A we get

$$A \cong M_{n_1}(D_1) \oplus M_{n_2}(D_2) \oplus ... \oplus M_{n_s}(D_s), \quad (2)$$

where $\sum_{i=1}^{s} n_i^2 \dim_{\mathbf{R}} D_i = n$ and D_i are algebras with a division over $\mathbf{R}$ for $i = 1,2,...,s$. Since A is a commutative algebra then $M_{n_i}(D_i)$ are commutative algebras. Therefore $n_i = 1$ for each $i = 1,2,...,s$ and by the theorem of Frobenius (Pontryagin 1986, Pontryagin 1987) it can be deduced that $D_i = \mathbf{R}$ or $D_i = \mathbf{C}$ for $i = 1,2,...,s$, i.e. (1) holds.

Definition. Let A be a real finite-dimensional commutative semisimple algebra and $\dim_{\mathbf{R}} A = n$ ($n \in N$). We call the number r_A of the direct addends $\mathbf{R}$ in the decomposition (1) a *real cardinality of* A.

Theorem 2. *If* A *is a real finite-dimensional commutative semisimple algebra and* $\dim_{\mathbf{R}} A = n$ *(*$n \in N$*), then the real finite-dimensional commutative semisimple algebra* B *is isomorphic to* A *as an* $\mathbf{R}$*-algebra if and only if* $\dim_{\mathbf{R}} B = n$ *and* $r_A = r_B$.

Proof. The necessity is obvious and the sufficiency is given by the fact that the power of the algebra A and the number of direct addends $\mathbf{R}$ in (1) determine A up to isomorphism.

Theorem 3. *Let* A *be a real finite-dimensional commutative semisimple algebra and* G *be a finite abelian group. Then the algebra* A *is isomorphic as an* $\mathbf{R}$*-algebra to the group algebra* $\mathbf{R}G$ *if and only if* $\dim_{\mathbf{R}} A = |G|$ *and the real cardinality* r_A *of* A *is equal to* $|G[2]|$.

Proof. Necessity. Let A be isomorphic as $\mathbf{R}$-algebra to the group algebra $\mathbf{R}G$. Then $\dim_{\mathbf{R}} A = \dim_{\mathbf{R}} \mathbf{R}G = |G|$. We shall prove that the real cardinality r_A of A (i.e. the real cardinality $r_{\mathbf{R}G}$ of $\mathbf{R}G$) is equal to $|G[2]|$. The group algebra $\mathbf{R}G$ by the condition of the theorem is semisimple. Then $\mathbf{R}G \cong \dot{\sum} \mathbf{R}Ge_\chi$, where e_χ are different minimum idempotents of $\mathbf{R}G$, which correspond to the characters χ of the group G. The real cardinality $r_{\mathbf{R}G}$ of $\mathbf{R}G$ is equal to the number of those characters $\chi : G \to \mathbf{R}^*$ for which $g\chi = \pm 1$ for each $g \in G$. Let $G = \langle g_1 \rangle \times ... \times \langle g_s \rangle \times H$ is the decomposition of G in direct product of primary groups where $\langle g_i \rangle$ are cyclic 2-groups ($i = 1,...,s$) and 2 does not divide $|H|$, i.e. $|G[2]| = 2^s$. For the direct factor H there is a single character χ_0 with the mentioned properties, namely $h\chi_0 = 1$ for each $h \in H$. For each of the direct factors $\langle g_i \rangle$ there are two different such characters χ_{i0} and χ_{i1}, namely $g_i\chi_{i0} = 1$ and $g_i\chi_{i1} = -1$. Therefore, the number of all characters χ of G with the property $g\chi = \pm 1$ for each $g \in G$ is $2^s = |G[2]|$. Thus $r_{\mathbf{R}G} = |G[2]|$, and from the isomorphism we get $r_A = |G[2]|$. Since the case $G = H$ is trivial, then the proof of the necessity is complete.

Sufficiency. Let $\dim_{\mathbf{R}} A = |G|$ and the real cardinality r_A of A is equal to $|G[2]|$. In order to prove that A is isomorphic as $\mathbf{R}$-algebra to the group algebra $\mathbf{R}G$ it is enough, according to Theorem 1, to prove that $\dim_{\mathbf{R}} A = \dim_{\mathbf{R}} \mathbf{R}G$ and that the real cardinalities of the two algebras are equal, i.e. $r_A = r_{\mathbf{R}G}$. The first condition $\dim_{\mathbf{R}} A = \dim_{\mathbf{R}} \mathbf{R}G$ can be obtained from $\dim_{\mathbf{R}} \mathbf{R}G = |G|$. The second condition holds, since in the necessity we proved that $r_{\mathbf{R}G} = |G[2]|$.

Note. Let G and $\overline{G}$ be finite abelian groups. We can give by using the condition of Theorem 3 the following necessary and sufficient condition for an isomorphism of the group algebras $\mathbf{R}G$ and $\mathbf{R}\overline{G}$:

The real group algebras $\mathbf{R}G$ *and* $\mathbf{R}\overline{G}$ *of the finite abelian groups* G *and* $\overline{G}$ *are isomorphic as* $\mathbf{R}$*-algebras if and only if* $|G| = |\overline{G}|$ *and* $|G[2]| = |\overline{G}[2]|$.

The last result is a partial case of the result of Berman and Bogdan (Berman, Bogdan 1977).

Theorem 4. *Let* A *be a real finite-dimensional commutative algebra. Then* A *is isomorphic as an* $\mathbf{R}$*-algebra to some real group algebra if and only if the following conditions are met:*

(i) A *is semisimple algebra;*

(ii) $r_A = 2^t$, *where* t *is non-negative integer;*

(iii) r_A *divides* $\dim_R A$.

Proof. Necessity. Let A be isomorphic as an $\mathbf{R}$-algebra to the group algebra $\mathbf{R}G$ for some group G. Since A is a finite-dimensional and commutative algebra, then G is a finite abelian group. The algebra $\mathbf{R}G$ by the theorem of Maschke (Pierce 1986, van der Waerden 1990, Lang 2002) is semisimple which implies that A is semisimple, i.e. (i) is fulfilled.

By Theorem 3, the equality $r_A = |G[2]|$ is fulfilled. Consequently $r_A = 2^t$ for some non-negative integer t. In this way (ii) is proved.

Since $|G[2]|$ divides $|G|$ where $|G| = \dim_{\mathbf{R}} A$ and $r_A = |G[2]|$ holds, then r_A divides $\dim_{\mathbf{R}} A$, i.e. (iii) is fulfilled. The necessity is proved.

Sufficiency. Let the conditions (i), (ii) and (iii) hold. The condition (i) and Theorem 1 imply that the decomposition (1) holds, i.e.

$$A \cong \mathbf{R} \oplus \ldots \oplus \mathbf{R} \oplus \mathbf{C} \oplus \ldots \oplus \mathbf{C},$$

where, by (ii), the real cardinality of A is $r_A = 2^t$. We denote $n = \dim_{\mathbf{R}} A$. Let G be an arbitrary abelian group of order n whose 2-component is decomposed in direct product of t cyclic groups. The existence of such group when $t \geq 1$ is given by conditions (ii) and (iii). When $t = 0$ we get $n = 1 + 2c_A$ where c_A is the number of the direct addends $\mathbf{C}$ in the decomposition (1) of A. As n is an odd integer, then each abelian group G of order n satisfies the condition for the 2-component. When we apply Theorem 3 to A and $\mathbf{R}G$ we get that $A \cong \mathbf{R}G$ as $\mathbf{R}$-algebras. The proof of the sufficiency is completed.

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